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On some inverse properties for univalent functions

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Abstract. The object of the present paper is to investigate some inverse properties for univalent functions in the open unit disk U. Starlikeness and convexity for functions in U are shown.

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n} \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Let S be the subclass of A consisting of functions $f(z)$ which are univalent in U. It is very famous as Bieberbach conjecture that if $f(z)\in S$, then

$$|a_{n}|\leq n \qquad (n=2,3,4,\ldots). \tag{1.2}$$

The equality holds true for the Koebe function $k(z)$ which given by

$$k(z)=\frac{z}{(1-e^{i\theta}z)^{2}} \qquad (\theta\in\mathbb{R}). \tag{1.3}$$

This Bieberbach conjecture was proved by de Branges [1].
In the present paper, we investigate some inverse properties for functions $f(z)$ belonging to the class S.
Let B denote the class of functions $f(z)$ of the form (1.1) which satisfy the coefficient inequalities (1.2). Recently, Kim and Nunokawa [2, Theorem 1] proved that if $f(z)\in B$, then $f(z)$ is univalent in $|z|<r_{0}$, where r_{0} is the unique solution of the equation

$$2r^{3}-6r^{2}+7r-1=0. \tag{1.4}$$

This result is sharp.

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2 Inverse properties

For the functions $f(z)$ belonging to the class B, we derive

Theorem 1. *If* $f(z) \in B$, *then*

$$\frac{2r^2 - 4r + 1}{(1-r)^2} \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1-r)^2} \tag{2.1}$$

for $|z| = r < 1$. *The result is sharp for* $f(z) = z/(1 - e^{i\theta} z)^2$.

Proof. Since $f(z) \in B$ satisfies (1.2), we have

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n| \, |z|^n$$

$$\leq |z| + \sum_{n=2}^{\infty} n \, |z|^n = \frac{r}{(1-r)^2} \tag{2.2}$$

for $|z| = r < 1$.
Therefore, $f(z)$ absolutely converges in U, and so, $f(z)$ is analytic in U.
On the other hand, we have

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| \, |z|^n$$

$$\geq r - \sum_{n=2}^{\infty} nr^n \geq \frac{(2r^2 - 4r + 1)r}{(1-r)^2} \tag{2.3}$$

for $|z| = r < 1$.

□

Remark 1. Theorem 1 shows that $|f(z)/z| > 0$ for $|z| < r_1 = \frac{2-\sqrt{2}}{2} \fallingdotseq 0.29289$. Thus Theorem 1 is sharp.

Next we show

Theorem 2. *If* $f(z) \in B$, *then* $f(z)$ *is univalent and starlike in* $|z| < r_2$, *where*

$$r_2 = \frac{1}{1+\sqrt{2}} \left(1 - \sqrt{\frac{e}{2e-1}} \right) \fallingdotseq 0.08998. \tag{2.4}$$

Proof. By means of Theorem 1, we have $|f(z)/z| > 0$ in $|z| < r_1 = (2-\sqrt{2})/2$, and therefore, $\log(f(z)/z)$ is harmonic in $|z| < r_1$.
From the harmonic function theory, we know that

$$\log\frac{f(z)}{z} = \frac{1}{2\pi}\int_0^{2\pi}\left(\log\left|\frac{f(\zeta)}{\zeta}\right|\right)\frac{\zeta+z}{\zeta-z}d\varphi, \tag{2.5}$$

where $\zeta = \rho e^{i\varphi}(0 \leq \varphi \leq 2\pi), z = re^{i\theta}(0 \leq \theta \leq 2\pi)$, and $0 \leq r < \rho \leq r_1 = (2-\sqrt{2})/2$.

By using the logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2\pi}\int_0^{2\pi}\left(\log\left|\frac{f(\zeta)}{\zeta}\right|\right)\frac{2\zeta z}{(\zeta-z)^2}d\varphi. \tag{2.6}$$

Because, we have

$$\frac{1}{(1-r)^2} < \frac{(1-r)^2}{2r^2-4r+1} \tag{2.7}$$

for $|z| = r < 1$, then, from Theorem 1 and (2.7), we derive

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1 - \frac{1}{2\pi}\int_0^{2\pi}\left(\max_{|\zeta|=\rho}\left|\log\left|\frac{f(\zeta)}{\zeta}\right|\right|\right)\frac{2\rho r}{\rho^2-2\rho\cos(\varphi-\theta)+r^2}d\varphi$$

$$\geq 1 - \frac{2\rho r}{\rho^2-r^2}\log\frac{(1-\rho)^2}{2\rho^2-4\rho+1}, \tag{2.8}$$

where $0 \leq r < \rho < r_1 = (2-\sqrt{2})/2$.
Putting $\rho = (1+\sqrt{2})r$, we have

$$\frac{2\rho r}{\rho^2-r^2}\log\left(\frac{\rho^2-2\rho+1}{2\rho^2-4\rho+1}\right) = \log\left(\frac{1}{2}+\frac{1}{4\left\{(1+\sqrt{2})r-1\right\}^2-2}\right) = 1. \tag{2.9}$$

Consequently, we see that (2.8) and (2.9) imply

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{2.10}$$

in $|z| < r_2$, where r_2 is the smallest positive root of the equation

$$\frac{1}{2}+\frac{1}{4\left\{(1+\sqrt{2})r-1\right\}^2-2} = e \tag{2.11}$$

or

$$r_2 = \frac{1}{1+\sqrt{2}}\left(1-\sqrt{\frac{e}{2e-1}}\right) \fallingdotseq 0.08998. \tag{2.12}$$

This completes the proof of Theorem 2. □

Remark 2. In the proof of Theorem 2, we put $\rho = (1+\sqrt{2})r$. But we don't prove that this is best or not. Therefore, Theorem 2 is not sharp.

From Theorem 2, we make

Corollary 1. *If a function $f(z)$ of the form (1.1) satisfies*

$$|a_n| \leq 1 \qquad (n = 2, 3, 4, \dots),$$

then $f(z)$ is univalent and convex in $|z| < r_2$.

Applying the same method as the proof of Theorem 2, we can obtain some routh results on the other cases, but we expect that someone get exact results.

References

[1] L. de Branges, *A proof of Bieberbach conjecture*, Acta Math.**154**(1985),137-152.

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